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COMMENT

Lattice two-point functions and conformal invariance

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Abstract. A new realization of the conformal algebra is studied which mimics the behaviour of a statistical system on a discrete albeit infinite lattice. The two-point function is found from the requirement that it transforms covariantly under this realization. The result is in agreement with explicit lattice calculations of the $(1+1)$-dimensional Ising model and the d-dimensional spherical model. A hard core is found which is not present in the continuum. For a semi-infinite lattice, profiles are also obtained.

Since the pioneering work of Polyakov (1970), conformal invariance has become a powerful tool in the description and understanding of critical phenomena, in particular for two-dimensional (2D) systems (Belavin *et al* 1984). For example, critical exponents and the n-point correlation functions at criticality can be exactly determined and a classification of the universality classes obtained, see for example Christe and Henkel (1993) and di Francesco *et al* (1997) for an introduction.

The basis of this theory is the assertion that there is a certain class of scaling operators, called (quasi)primary, which transform covariantly under (global) conformal transformations. For notational simplicity, we shall work in a two-dimensional setting throughout, but the generalization to higher spatial dimensions will be immediate. For the same reason, we restrict attention to scalar scaling operators. The generators of the conformal algebra may be written in the form $\ell_n = -\hat{X}^{n+1}\hat{P}$, where $\hat{X}$ and $\hat{P}$ are operators which satisfy the commutation relation $[\hat{P}, \hat{X}] = 1$ (we neglect here the terms involving the central charge). Usually, an underlying continuum theory is assumed and then the choice of realization

$$\hat{X} = z \qquad \hat{P} = \frac{\partial}{\partial z} \tag{1}$$

is natural. However, other realizations are perfectly possible. For example, in the context of 2D turbulence, new realizations of the conformal algebra which yield logarithmic two-point functions are needed (Rahimi Tabar *et al* 1997, Flohr 1996). Here, we shall consider yet another realization which mimics the behaviour of a system defined on a discrete lattice. Field theories on a discrete (spacetime) lattice are presently under active study, for example Kauffman and Noyes (1996), Winitzki (1997) and Faddeev and Volkov (1997) and references therein. Following an idea exploited in connection with Schrödinger invariance (Henkel and Schütz 1994), we shall work with

$$\hat{X} = \frac{1}{\cosh(\frac{1}{2}a\partial_z)} z \qquad \hat{P} = \frac{2}{a}\sinh\left(\tfrac{1}{2}a\partial_z\right) \tag{2}$$

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where a is a free parameter which we shall interpret as a lattice constant. Comparing the realizations (1) and (2), we see that the usual infinitesimal translation operator $\ell_{-1}^{(a=0)} f(z) = -\partial_z f(z)$ is replaced by the symmetric discrete difference

$$\ell_{-1}^{(a)} f(z) = -\partial_z^a f(z) := -\frac{1}{a}\left[f\left(z + \frac{a}{2}\right) - f\left(z - \frac{a}{2}\right)\right].$$

For a better geometric understanding of these transformations, we briefly consider the finite transformations generated by the ℓ_n. These are given by the functions $S(\epsilon, z) = \exp(\epsilon \ell_n) z$, together with the initial condition $S(0, z) = z$. They may be obtained by solving the equation

$$\frac{\partial S}{\partial \epsilon}(\epsilon, z) = \ell_n S(\epsilon, z). \tag{3}$$

In the case of the continuum realization (1), one has $S(\epsilon, z) = z - \epsilon$ for translations ($n = -1$) and $S(\epsilon, z) = z\,\mathrm{e}^{-\epsilon}$ for dilatations ($n = 0$). However, it can be easily checked that the same solutions of (3) also apply for the 'lattice' realization (2). The lattice is mapped onto itself, provided ϵ is restricted to discrete values, for example $\epsilon = ma$ (m integer) for translations and $\epsilon = -\ln p$ (p a positive integer) for dilatations. Discrete dilatations also appear naturally in connection with the generators of (multi)fractal sets (Erzan and Eckmann 1997) and the critical behaviour of certain aperiodic quantum chains (Karevski and Turban 1996), see also Sornette (1997).

We now focus on the calculation of the two-point correlation function $F(r_1, t_1; r_2, t_2) = \langle \phi_1(r_1, t_1)\phi_2(r_2, t_2)\rangle$. We assume that the 'space' direction is discretized and the 'time' direction is continuous, that the system is infinite in both directions and invariant under the global conformal group generated by $\ell_{\pm 1,0}$ and $\bar{\ell}_{\pm 1,0}$. Here the ϕ_i are (scalar) scaling operators of scaling dimensions x_i. Following the procedure which is standard in the continuum (Polyakov 1970), we require F to transform covariantly under the 'lattice' realization (2).

From translation invariance in both space and time directions, it appears physically obvious to take $F(r_1, t_1; r_2, t_2) = F(r, t)$ with $r = r_2 - r_1$ and $t = t_2 - t_1$, since this agrees with the continuum limit $a \to 0$ (more precisely, the equation $(\partial_{r_1}^a + \partial_{r_2}^a)F = 0$ has two solutions, only one of which has the expected $a \to 0$ limit (Henkel and Schütz 1994)). From rotational invariance†, F should satisfy the differential equation $\mathrm{i}(\ell_0 - \bar{\ell}_0)F(r, t) = \left[-t\partial_r^a + (\cosh(\frac{1}{2}a\partial_r))^{-1} r\partial_t\right] F(r, t) = 0$. Fourier transformation gives

$$\left[\frac{2}{a}\sin\left(\frac{a}{2}k\right)\partial_\omega - \frac{\omega}{\cos\left(\frac{1}{2}ak\right)}\partial_k\right]\widetilde{F}(k, \omega) = 0 \tag{4}$$

where $\widetilde{F}(k, \omega)$ is the Fourier transform of $F(r, t)$. The general solution of (4) is $\widetilde{F}(k, \omega) = \widetilde{\mathcal{F}}(\omega^2 + [(2/a)\sin(ak/2)]^2)$, where $\mathcal{F}$ is an arbitrary function (in the continuum limit, $a \to 0$, one recovers the usual form $\widetilde{\mathcal{F}}(\omega^2 + k^2)$).

Here a comment is in order. The argument of the function $\mathcal{F}$ is related to the Casimir operator C of the subalgebra generated by translations and rotations. Dilatation invariance fixes $C = 0$, or equivalently yields the dispersion relations $E^2 = k^2$ for the continuum and $E^2 = 4/a^2 \sin^2(ak/2)$ for the 'lattice' realization. Using the 'lattice' realization (2) automatically leads to the lattice dispersion relation. We can, therefore, expect the hypothesis of covariance under the 'lattice' realization (2) to apply to systems which satisfy that dispersion relation on a discrete lattice with lattice constant a. Model examples for these

† For brevity, this infinitesimal transformation is called rotation by analogy with the real infinitesimal rotation to which it reduces for $a \to 0$.

situations are provided by systems with an underlying field theory satisfying this dispersion relation, see later.

Finally, covariance under dilatations, generated by $\ell_0 + \bar{\ell}_0$, gives in Fourier space

$$\left[\partial_\omega \omega + \frac{2}{a}\frac{1}{\cos(\frac{1}{2}ak)}\partial_k \sin\left(\frac{ak}{2}\right)\right]\widetilde{F}(k,\omega) = (x_1 + x_2)\,\widetilde{F}(k,\omega) \tag{5}$$

(where the differential operators act on everything to the right) and the solution is

$$\widetilde{F}(k,\omega) = \widetilde{F}_0\left[\omega^2 + \left(\frac{2}{a}\sin\left(\frac{ak}{2}\right)\right)^2\right]^{(x_1+x_2-2)/2}$$

where $\widetilde{F}_0$ is a constant. Finally, a straightforward calculation shows (see Henkel and Schütz 1994) that covariance under $\ell_1, \bar{\ell}_1$ merely adds the extra condition $x_1 = x_2 =: x_\phi$. The two-point function is

$$F(r,t) = A\,\delta_{x_1,x_2} a^{-2x_\phi}\int_0^\infty \mathrm{d}v\, v^{-x_\phi - \frac{1}{2}}\exp\left(-\frac{\tau^2}{v} - v\right) I_\rho(v) \tag{6}$$

with $\tau^2 = t^2/(2a^2)$, $\rho = r/a$ is a positive integer, $I_\rho(v)$ is a modified Bessel function and A is a normalization constant. In fact, this can be written explicitly in terms of hypergeometric functions ${}_1F_2$ (Prudnikov *et al* 1992) and reads $F(r,t) = A\delta_{x_1,x_2}a^{-2x_\phi}\Psi$, where

$$\Psi = \frac{2^{x_\phi - \frac{1}{2}}}{\sqrt{\pi}}\frac{\Gamma(\rho + \frac{1}{2} - x_\phi)\Gamma(x_\phi)}{\Gamma(\rho + \frac{1}{2} + x_\phi)}\,{}_1F_2(x_\phi; x_\phi + \tfrac{1}{2} - \rho, x_\phi + \tfrac{1}{2} + \rho; 2\tau^2)$$
$$+2^{-\rho}\tau^{2(\rho + \frac{1}{2} - x_\phi)}\frac{\Gamma(x_\phi - \frac{1}{2} - \rho)}{\Gamma(1+\rho)}\,{}_1F_2(\rho + \tfrac{1}{2}; \tfrac{3}{2} - x_\phi + \rho, 1 + 2\rho; 2\tau^2). \tag{7}$$

A particularly simple form is found for the correlation purely in the 'space' direction, along the discrete lattice, that is $t = 0$

$$F(r,0) = F_0\delta_{x_1,x_2}a^{-2x_\phi}\frac{\Gamma(\rho + \frac{1}{2} - x_\phi)}{\Gamma(\rho + \frac{1}{2} + x_\phi)}. \tag{8}$$

In the limit $\rho = r/a \gg 1$ this reduces to the standard continuum form $F(r) = F_0 r^{-2x_\phi}$, as it should be. We also remark that the power-law dependence on a is consistent with the interpretation of a as a length.

A new feature of the 'lattice' realization is a *hard core* in the two-point function. This effect is absent in the continuum limit $a \to 0$. To see this, note that for $\frac{1}{2}+n \leqslant x_\phi < \frac{1}{2}+n+1$ with $n = 0, 1, 2, \ldots,$ the position ρ_s of the first singularity of $F(r,0)$ is in the interval $n \leqslant \rho_s < n+1$ (for $x_\phi < \frac{1}{2}$, there is no singularity of $F(r,0)$ for positive r). For $\rho \leqslant \rho_s$, $F(r,0)$ is no longer defined. The system behaves as if the scaling operator ϕ would contain a hard sphere of diameter $\mathcal{D} = (x_\phi - \frac{1}{2})a$, thus preventing an approach closer than $\rho = \rho_s \geqslant n$. For $x_\phi \leqslant \frac{1}{2}$, the operator ϕ does not contain such a hard sphere at all. For larger values of x_ϕ, a hard sphere is always present, but for $x_\phi < \frac{3}{2}$ its diameter $\mathcal{D}$ is still smaller than the lattice constant a. On the other hand, the scaling operator corresponding to the order parameter σ should be as local as possible. If we take as locality criterion the absence of a hard sphere at all, this suggests that its scaling dimension $x_\sigma \leqslant \frac{1}{2}$, but if it is enough that two order parameter operators can be defined on neighbouring sites with distance a, equation (8) suggests that $x_\sigma \leqslant \frac{3}{2}$.

As already noted, equation (6) should be correct for models with an underlying free field theory satisfying the dispersion relation $E = (2/a)|\sin(ak/2)|$. Examples of these are the spherical model and the even sector of the $(1+1)$-dimensional Ising model. Consider first a d-dimensional spherical model, continuous in the $d-1$ directions labelled by a vector position $\boldsymbol{r}_\perp$ and discrete in the direction $r_\|$, with $a=1$. The spin–spin correlation function at the critical temperature T_c is (Berlin and Kac 1952)

$$C(r_\|, \boldsymbol{r}_\perp) = \frac{k_B T_c}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} d\phi_1 \ldots d\phi_d \frac{\cos(\boldsymbol{r}_\perp \Phi_\perp + r_\| \phi_\|)}{2Jd - 2J(\cos\phi_1 + \cdots + \cos\phi_d)} \tag{9}$$

where J is the coupling constant and $\Phi_\perp$ is the vector $(\phi_1, \ldots, \phi_{d-1})$. Using the expansion $\cos\phi_i \simeq 1 - \phi_i^2/2$ in the $d-1$ continuous directions, we obtain in a standard fashion, for $2 < d < 4$,

$$C(r_\|, \boldsymbol{r}_\perp) = \frac{k_B T_c}{2J(2\pi)^{(d-1)/2}} \int_0^\infty du\, u^{-\frac{1}{2}(d-2)-\frac{1}{2}} \exp\left(-\frac{r_\perp^2}{2u} - u\right) I_{r_\|}(u) \tag{10}$$

where $r_\perp^2 = \sum_{i=1}^{d-1} r_i^2$ and $r_\|$ are both integers. With the identification $x_\phi = x_\sigma = \beta/\nu = (d-2)/2$ for the spherical model, we obtain exactly the same form as equation (6).

We turn now to the $(1+1)$-dimensional Ising model. In fact, a test of equation (8) is easier to perform in the strongly anisotropic limit, which implies a continuum limit in the 'time' direction while keeping the 'space' direction discrete and maps the 2D Ising model onto the Ising quantum chain in a transverse field. Then the two-point correlation function for the energy density operator ε is (Pfeuty 1970)

$$C_\varepsilon(\rho, 0) = \frac{1}{4\rho^2 - 1} \tag{11}$$

while from equation (8), with $x_\varepsilon = 1$ for the scaling dimension of the energy density, we expect up to normalization $F_\varepsilon(\rho, 0) = 1/(4\rho^2 - 1)$. On the other hand, the spin–spin correlation function is not of the form (8), but for $\rho \gg 1$ one has (Pfeuty 1970)

$$C_\sigma(\rho, 0) \simeq C_{\sigma,0}\, \rho^{-1/4} \left[1 - \frac{1}{64}\rho^{-2}\right] \tag{12}$$

while our formula (8), with $x_\sigma = 1/8$, leads to $F_\sigma(\rho, 0) \sim \rho^{-1/4}[1 - 1/(48\rho^2)]$. This discrepancy between odd and even sectors of the Ising model is not surprising. Indeed, *only* the even sector alone is described by a free fermionic field. However, the order parameter intertwines between the even and odd symmetry sectors of the model and does *not* correspond to a free field with a lattice dispersion relation $E = (2/a)|\sin(ak/2)|$, in contrast to the requirements in the derivation of (8).

Finally, we consider briefly the case of a semi-infinite geometry, with the boundary perpendicular to the discretization axis, so that we have for the position coordinates $-\infty < \boldsymbol{r}_\perp < \infty$ and $0 < r_\| < \infty$. We consider in this case a non-vanishing profile of a scaling operator ϕ, $C(\boldsymbol{r}_\perp, r_\|) = \langle \phi(\boldsymbol{r}_\perp, r_\|) \rangle$. From translation invariance along the perpendicular directions, $C(\boldsymbol{r}_\perp, r_\|) = F(r_\|)$. Using a Laplace transformation

$$F(r_\|) = \int_0^\infty e^{-r_\| s} f(s)\, ds \tag{13}$$

covariance of $F(r_\|)$ under dilatations (generated by ℓ_0) leads to, for $x_\phi > 0$,

$$\frac{2}{a} \frac{1}{\cosh(\frac{1}{2}as)} \partial_s \left(\sinh\left(\frac{as}{2}\right) f(s)\right) = x_\phi f(s) \tag{14}$$

with the solution $f(s) = A[2a^{-1}\sinh(\frac{1}{2}as)]^{x_\phi - 1}$, where A is a constant. We find

$$F(r_\|) = F_0\, a^{-x_\phi} \frac{\Gamma((r_\|/a) + \frac{1}{2} - \frac{1}{2}x_\phi)}{\Gamma((r_\|/a) + \frac{1}{2} + \frac{1}{2}x_\phi)} \tag{15}$$

which is exactly the same result as in equation (8), but for the replacement of x_ϕ by $x_\phi/2$. In the continuum limit, $r_\parallel/a \gg 1$, we recover the Fisher and de Gennes (1978) result $F(r_\parallel) \sim r_\parallel^{-x_\phi}$. We observe an analogous hard core effect as found for the two-point function for the infinite system.

Summarizing, we have studied a new realization of the conformal algebra which mimics a system defined on a discrete lattice. From the covariance under global conformal transformations, the two-point function was obtained and found to be in agreement with explicit results in the spherical model and the even sector of the $(1+1)$-dimensional Ising model. It displays a hard core not present in the continuum limit.

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